Galois Connections on Lattices

Dennis Sprokholt

August 2022

1. Introduction

In my opinion, Galois Connections are often explained, either:

- *too general* – where it relies on *group theory*, *algebraic structures*, or *category theory* (none of which I'm fluent in), or
- *too specific* – where it *only* addresses its *application* to abstract interpretation, failing to explain why they are *the* solution.

In this note, I explain Galois Connections *generally on lattices*, and work toward their *specialization* to abstract interpretation.

2. Lattices

We shall work our way up from lattices to Galois connections. We consider *order-theoretic* definitions of lattices, as opposed to *algebraic* definitions[1], which seems most suitable for our purposes. Let's start with a property:

Definition 1: Join ($\vee$)

The *join* '$x \vee y$' for *some* $x, y \in A$ is the *lowest* element in poset A that is greater than both x and y. Which we formally define as:

- *Greater than* x: $x \;\leq\; x \vee y$
- *Greater than* y: $y \;\leq\; x \vee y$
- *Tight*: $\forall z \;.\; x \leq z \;\wedge\; y \leq z \;\;\rightarrow\;\; (x \vee y) \leq z$

We denote 'join' with $\vee$ to avoid confusion with logical disjunction $\vee$, which may occur in the same sentence. An example of $\vee$ is:

[1] These definitions are (mostly) equivalent. The algebraic definitions derives $\leq$ from $\vee$ / $\wedge$.

Example 1: Join

Consider poset $(\mathcal{P}(\mathbb{N}), \subseteq)$ with join $\cup$. We take:

- $x = \{1, 4\}$
- $y = \{4, 6\}$

Here, $x \ \cup \ y = \{1, 4\} \cup \{4, 6\} = \{1, 4, 6\}$. Then it satisfies:

- *Greater than* x: $\{1, 4\} \ \subseteq \ \{1, 4, 6\}$
- *Greater than* y: $\{4, 6\} \ \subseteq \ \{1, 4, 6\}$
- *Tight*: $\forall z \ . \ \{1, 4\} \subseteq z \ \wedge \ \{4, 6\} \subseteq z \quad \rightarrow \quad \{1, 4\} \cup \{4, 6\} \ \subseteq \ z$

The "Tight" property is a little more complicated. Intuitively, it says that there *exists no* z that is greater than both $\{1, 4\}$ and $\{4, 6\}$, while smaller than $\{1, 4, 6\}$.

. . . .

Suppose we had picked an "over-approximating union" $\tilde{\cup}$ as join, where

$$\{1, 4\} \ \tilde{\cup} \ \{4, 6\} \ \triangleq \ \{1, 4, 6, 42\}$$

$\tilde{\cup}$ is *not* tight. Pick $z = \{1, 4, 6\}$, then:

$$\{1, 4\} \subseteq \{1, 4, 6\} \quad \text{and} \quad \{4, 6\} \subseteq \{1, 4, 6\}$$

However,

$$\{1, 4\} \ \tilde{\cup} \ \{4, 6\} \ \not\subseteq \ \{1, 4, 6\}$$

Hence, "tight" ensures that our join is the *least* upper bound.

The *dual* of join ($\vee$) is *meet* ($\wedge$):

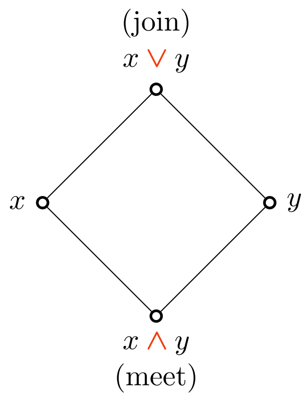
Definition 2: Meet ($\wedge$)

The *meet* '$x \wedge y$' for *some* $x, y \in A$ is the *greatest* element in poset A that is less than both x and y. Which we formally define as:

- *Less than* x: $x \wedge y \ \leq \ x$
- *Less than* y: $x \wedge y \ \leq \ y$
- *Tight*: $\forall z \ . \ z \leq x \ \wedge \ z \leq y \quad \rightarrow \quad z \leq (x \wedge y)$

We can visualize these definitions in a Hasse Diagram:

Figure 1: $\vee$ and $\wedge$ Hasse Diagram



Now, the definition of lattices is quite simple.

Definition 3: $\vee$-semilattice (join-semilattice)

A poset $(A, \leq)$ is a $\vee$-semilattice iff a $\vee$ exists for *every pair* of elements in A.

Again, its dual is:

Definition 4: $\wedge$-semilattice (meet-semilattice)

A poset $(A, \leq)$ is a $\wedge$-semilattice iff a $\wedge$ exists for *every pair* of elements in A.

We can combine these definitions as:

Definition 5: Lattice

A poset $(A, \leq)$ is a lattice iff both a $\vee$ and $\wedge$ exist for *every pair* of elements in A.

For the remaining sections, only the semilattices are relevant.

2.1. Galois Connections

With semilattices established, we can apply Galois Connections. The typical definition of a (monotone[2]) Galois Connection is:

Definition 6: Galois Connection

Given two posets $(A, \preccurlyeq)$ and $(B, \sqsubseteq)$ we define the Galois Connection (f, g) where

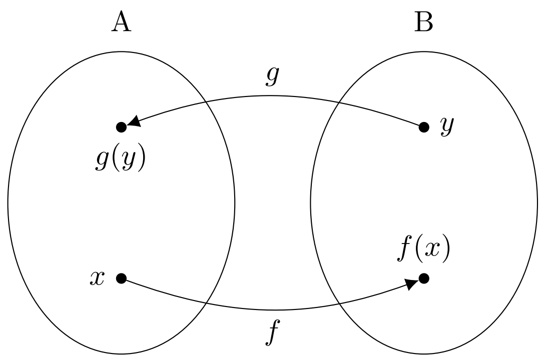
$$f : A \to B$$
$$g : B \to A$$

Then, f and g satisfy (for all $x \in A$ and $y \in B$):

$$f(x) \ \sqsubseteq \ y \quad \Leftrightarrow \quad x \ \preccurlyeq \ g(y)$$

We can visualize this definition as:

Figure 2: Galois Connection



Note that both f and g are monotone functions, which follows from Definition 6. See also Proof 1 in Appendix A.

[2]Galois Connections can also be antitone, which "flips" the order. Those are irrelevant here.

The definition of Galois Connections merely requires posets, which need not necessarily be lattices. We demonstrate that, on lattices, f and g *uniquely* determine each other.

Definition 7: Derived g

Given two posets $(A, \preccurlyeq)$ and $(B, \sqsubseteq)$ where A is a $\vee$-semilattice with join $\vee$, and given monotone function $f : A \to B$. Then:

$$g(y) \quad \triangleq \quad \bigvee \{ \ z \mid f(z) \sqsubseteq y \ \}$$

Similarly:

Definition 8: Derived f

Given two posets $(A, \preccurlyeq)$ and $(B, \sqsubseteq)$ where B is a $\wedge$-semilattice with meet $\wedge$, and given monotone function $g : B \to A$. Then:

$$f(x) \quad \triangleq \quad \bigwedge \{ \ z \mid x \preccurlyeq g(z) \ \}$$

Of course, we need to demonstrate that these definitions actually produce Galios Connections (as given in Definition 6). For that, see Proof 2 and Proof 3 in Appendix A.

Interestingly, these definitions enforce that *either* A is a $\vee$-semilattice, *or* B is a $\wedge$-semilattice. (While both are posets)

3. Abstract Interpretation

Now we can *specialize* our Galois Connections on lattices to *abstract interpretation*. In abstract interpretation we consider our *concrete domain* $\mathcal{C}$ (with partial order $\preccurlyeq$) and our *abstract domain* $\mathcal{A}$ (with partial order $\sqsubseteq$). We then define our surjective[3] *abstraction* function:

$$\alpha : \mathcal{C} \to \mathcal{A}$$

and our injective[4] *concretization* function:

$$\gamma : \mathcal{A} \to \mathcal{C}$$

These form a Galois connection (α, γ). Which thus means (for all $x \in \mathcal{C}$ and $y \in \mathcal{A}$):

$$\alpha(x) \sqsubseteq y \quad \Leftrightarrow \quad x \preccurlyeq \gamma(y)$$

That's it. We established how both functions of a Galois Connection are related, and briefly framed them in *abstract interpretation*.

That was all I considered missing in other explanations of abstract interpretation, and have now discovered. For other details on abstract interpretation, there are better resources in existence.

[3] surjective: $\forall y \in \mathcal{A} \ . \ \exists x \in \mathcal{C} \ . \ \alpha(x) = y$

[4] injective: $\forall x, y \in \mathcal{A} \ . \ \gamma(x) = \gamma(y) \ \ \to \ \ x = y$

A. Proofs

We consider f from Definition 6.

Proof 1: f is monotone

For Galois Connection (f, g) we demonstrate monotonicity of f (for all $x, y \in A$):

$$x \preccurlyeq y \quad \rightarrow \quad f(x) \sqsubseteq f(y)$$

Lemma 1. Because $f(x) \sqsubseteq f(y) \Rightarrow x \preccurlyeq g(f(y))$ by (Def 6) we know:

$$\forall x \ . \ x \preccurlyeq g(f(x))$$

If for any $x, y \in A$ we know $x \preccurlyeq y$, and by Lemma 1 we know $y \preccurlyeq g(f(y))$, then $x \preccurlyeq g(f(y))$. By the $\Leftarrow$ of (Def 6) we know $f(x) \sqsubseteq f(y)$. Hence, f is monotonic.

The proof for monotonicity of g is similar.

Proof 2: Definition 7 establishes a Galois Connection

We show both implications hold of:

$$f(x) \sqsubseteq y \quad \Leftrightarrow \quad x \preccurlyeq g(y)$$

. . . .

Given (A) $f(x) \sqsubseteq y$.

Lemma 2. From the definition of $\vee$ (for all x, $\vee$-semilattice X):

$$x \in X \quad \Rightarrow \quad x \ \preccurlyeq \ \bigvee X$$

We know $x \in \{z \mid z \preccurlyeq x\}$, so by Lemma 2: $x \preccurlyeq \bigvee\{z \mid z \preccurlyeq x\}$. Then:

$$\begin{aligned}
x &\preccurlyeq \bigvee\{z \mid z \preccurlyeq x\} && [\ f \text{ monotonic }] \\
&\preccurlyeq \bigvee\{z \mid f(z) \sqsubseteq f(x)\} && [\ \text{by transitivity of } \sqsubseteq \text{ with (A) }] \\
&\preccurlyeq \bigvee\{z \mid f(z) \sqsubseteq y\} && [\ \text{def } g(y)\ ] \\
&\preccurlyeq g(y)
\end{aligned}$$

. . . .

Given (B) $x \preccurlyeq g(y)$.

Lemma 3. Because $\vee$ is tight and f is monotonic[5] (for all y):

$$f(\bigvee\{z \mid f(z) \sqsubseteq y\}) \sqsubseteq y$$

We know by (B):

$$\begin{aligned}
& x \preccurlyeq g(y) && [\ \text{def } g(y)\ ] \\
\Leftrightarrow\quad & x \preccurlyeq \bigvee\{z \mid f(z) \sqsubseteq y\} && [\ f \text{ is monotonic }] \\
\Rightarrow\quad & f(x) \sqsubseteq f(\bigvee\{z \mid f(z) \sqsubseteq y\}) && [\ \text{by Lemma 3 (and transitivity of } \sqsubseteq)\ ] \\
\Rightarrow\quad & f(x) \sqsubseteq y
\end{aligned}$$

. . . .

Hence, Definition 7 establishes a Galois Connection.

[5] I know. This brushes over several details.

Proof 3: Definition 8 establishes a Galois Connection

We show both implications hold of:

$$f(x) \sqsubseteq y \quad \Leftrightarrow \quad x \preccurlyeq g(y)$$

. . . .

Given (A) $f(x) \sqsubseteq y$.

Lemma 4. Because $\wedge$ is tight and g is monotonic (for all x):

$$x \preccurlyeq g(\wedge\{z \mid x \preccurlyeq g(z)\})$$

We know by (A):

$$\begin{array}{lll} & f(x) \sqsubseteq y & [\text{ def } f(x) \text{ }] \\ \Leftrightarrow & \bigwedge\{z \mid x \preccurlyeq g(z)\} \sqsubseteq y & [\text{ } g \text{ is monotonic }] \\ \Rightarrow & g(\bigwedge\{z \mid x \preccurlyeq g(z)\}) \preccurlyeq g(y) & [\text{ by Lemma 4 (and transitivity of } \preccurlyeq) \text{ }] \\ \Rightarrow & x \preccurlyeq g(y) & \end{array}$$

. . . .

Given (B) $x \preccurlyeq g(y)$.

Lemma 5. From the definition of $\wedge$ (for all x, $\wedge$-semilattice X):

$$x \in X \quad \Rightarrow \quad \bigwedge X \sqsubseteq x$$

We know $y \in \{z \mid y \sqsubseteq z\}$, so by Lemma 5: $\bigwedge\{z \mid y \sqsubseteq z\} \sqsubseteq y$. Then:

$$\begin{array}{lll} & \bigwedge\{z \mid y \sqsubseteq z\} \sqsubseteq y & [\text{ } g \text{ is monotonic }] \\ \Rightarrow & \bigwedge\{z \mid g(y) \preccurlyeq g(z)\} \sqsubseteq y & [\text{ by transitivity of } \preccurlyeq \text{ with (B) }] \\ \Rightarrow & \bigwedge\{z \mid x \preccurlyeq g(z)\} \sqsubseteq y & [\text{ def } f(x) \text{ }] \\ \Leftrightarrow & f(x) \sqsubseteq y & \end{array}$$

. . . .

Hence, Definition 8 establishes a Galois Connection.